

COMPLEX NUMBERSINTRODUCTION

Consider the quadratic equation $z^2 - 4z + 13 = 0$.

This equation must be solved using the quadratic formula with $a = 1$, $b = -4$ and $c = 13$.

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 13}}{2 \times 1} \\ &= \frac{4 \pm \sqrt{-36}}{2} \end{aligned}$$

The quadratic equation has no real roots since $b^2 - 4ac < 0$. The non-real roots of this equation can be found as follows.

Now $\sqrt{-36} = \sqrt{36 \times (-1)} = \sqrt{36} \times \sqrt{-1} = 6i$, where $i = \sqrt{-1}$.

$$\text{Then } z = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Hence $z = 2 + 3i$ or $z = 2 - 3i$.

The roots above are known as **complex numbers**.

In general, a complex number is of the form $x + yi$, where x and y are real numbers and $i = \sqrt{-1}$.

Note that since $i = \sqrt{-1}$, this means that $i^2 = -1$.

i is known as an imaginary number. Although complex numbers are imaginary numbers, the use of complex number methods is essential for many branches of mathematics.

REAL AND IMAGINARY PARTS

Let $z = x + yi$ be a general complex number.

x is known as the **real part** of z and y is known as the **imaginary part** of z .

We write $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

$$\text{Re}(-2 + 5i) = -2 \quad \text{and} \quad \text{Im}(-2 + 5i) = 5$$

$$\text{Re}(-4i) = 0 \quad \text{and} \quad \text{Im}(-4i) = -4$$

EQUAL COMPLEX NUMBERS

Two complex numbers are equal if and only if both the real and imaginary parts of the complex numbers are equal.

$$a + bi = c + di \quad \Leftrightarrow \quad a = c \quad \text{and} \quad b = d$$

If we know that two complex numbers are equal, we can **equate the real and imaginary parts** of the complex numbers. The following example illustrates this principle.

Worked Example

Given that $x + 2yi = 3 + (x + 1)i$, where x and y are real numbers, find the values of x and y .

Solution

$$x + 2yi = 3 + (x + 1)i$$

$$\text{Equating real parts} \quad \Rightarrow \quad x = 3$$

$$\text{Equating imaginary parts} \quad \Rightarrow \quad 2y = x + 1$$

$$\Rightarrow \quad 2y = 4$$

$$\Rightarrow \quad y = 2$$

Hence $x = 3$ and $y = 2$.

ADDITION, SUBTRACTION AND MULTIPLICATION OF COMPLEX NUMBERS

Let $z = 4 + 3i$ and $w = 1 - 2i$.

$$\begin{aligned} z + w &= (4 + 3i) + (1 - 2i) \\ &= 4 + 3i + 1 - 2i \\ &= 5 + i \end{aligned}$$

$$\begin{aligned} z - w &= (4 + 3i) - (1 - 2i) \\ &= 4 + 3i - 1 + 2i \\ &= 3 + 5i \end{aligned}$$

$$\begin{aligned} zw &= (4 + 3i)(1 - 2i) \\ &= 4 - 5i - 6i^2 \\ &= 4 - 5i + 6 \quad [\text{since } i^2 = -1] \\ &= 10 - 5i \end{aligned}$$

$$\begin{aligned} z^2 &= (4 + 3i)^2 \\ &= (4 + 3i)(4 + 3i) \\ &= 16 + 24i + 9i^2 \\ &= 16 + 24i - 9 \\ &= 7 + 24i \end{aligned}$$

COMPLEX CONJUGATES

Given a complex number $z = x + yi$, where x and y are real numbers, the complex conjugate of z is denoted by \bar{z} and is defined as $\bar{z} = x - yi$.

$$z = x + yi \quad \Rightarrow \quad \bar{z} = x - yi$$

$$z = 1 + 2i \quad \Rightarrow \quad \bar{z} = 1 - 2i$$

$$z = 5 - 3i \quad \Rightarrow \quad \bar{z} = 5 + 3i$$

$$w = 4i \quad \Rightarrow \quad \bar{w} = -4i$$

$$z = 3 \quad \Rightarrow \quad \bar{z} = 3$$

Worked Example

Given the equation $z + 2i\bar{z} = 8 + 7i$ for the complex number z , express z in the form $a + ib$.

Solution

Let $z = a + ib$, where a and b are real numbers.

Then $\bar{z} = a - ib$.

$$\begin{aligned} z + 2i\bar{z} = 8 + 7i &\Rightarrow a + ib + 2i(a - ib) = 8 + 7i \\ &\Rightarrow a + ib + 2ia - 2i^2b = 8 + 7i \\ &\Rightarrow a + ib + 2ia + 2b = 8 + 7i \quad [\text{since } i^2 = -1] \\ &\Rightarrow (a + 2b) + i(b + 2a) = 8 + 7i \end{aligned}$$

$$\text{Equating real parts} \quad \Rightarrow \quad a + 2b = 8$$

$$\text{Equating imaginary parts} \quad \Rightarrow \quad 2a + b = 7$$

Solving these equations simultaneously gives $a = 2$ and $b = 3$.

Hence $z = 2 + 3i$.

DIVISION OF COMPLEX NUMBERS

To divide two complex numbers in the form of a quotient, **multiply both the numerator and denominator by the complex conjugate of the denominator**. This will change the denominator into a real number and the quotient can be expressed as a complex number. The following examples illustrate this method.

Worked Example 1

Express $\frac{8+i}{3+2i}$ in the form $x + yi$, where x and y are real numbers.

Solution

To simplify $\frac{8+i}{3+2i}$, multiply both the numerator and the denominator by $3-2i$.

$$\begin{aligned}\frac{8+i}{3+2i} &= \frac{(8+i)(3-2i)}{(3+2i)(3-2i)} \\ &= \frac{24-13i-2i^2}{9-4i^2} \\ &= \frac{24-13i+2}{9+4} \\ &= \frac{26-13i}{13} \\ &= 2-i\end{aligned}$$

Worked Example 2

Express $\frac{1-7i}{4-3i}$ in the form $x + yi$, where x and y are real numbers.

Solution

To simplify $\frac{1-7i}{4-3i}$, multiply both the numerator and the denominator by $4+3i$.

$$\begin{aligned}\frac{1-7i}{4-3i} &= \frac{(1-7i)(4+3i)}{(4-3i)(4+3i)} \\ &= \frac{4-25i-21i^2}{16-9i^2} \\ &= \frac{4-25i+21}{16+9} \\ &= \frac{25-25i}{25} \\ &= 1-i\end{aligned}$$

Miscellaneous Example

Find the two complex numbers z for which $z^2 = 5 - 12i$.

Solution

Let $z = x + yi$, where x and y are real numbers.

$$\begin{aligned}z^2 &= (x + yi)(x + yi) \\ &= x^2 + 2xyi + y^2i^2 \\ &= x^2 + 2xyi - y^2 \quad [\text{since } i^2 = -1] \\ &= (x^2 - y^2) + 2xyi\end{aligned}$$

$$z^2 = 5 - 12i \Rightarrow (x^2 - y^2) + 2xyi = 5 - 12i$$

$$\text{Equating real parts} \Rightarrow x^2 - y^2 = 5 \quad \dots(1)$$

$$\begin{aligned}\text{Equating imaginary parts} &\Rightarrow 2xy = -12 \\ &\Rightarrow y = \frac{-12}{2x} \\ &\Rightarrow y = -\frac{6}{x} \quad \dots(2)\end{aligned}$$

$$\begin{aligned}\text{Sub. (2) into (1):} \quad &x^2 - y^2 = 5 \\ \Rightarrow &x^2 - \left(-\frac{6}{x}\right)^2 = 5 \\ \Rightarrow &x^2 - \frac{36}{x^2} = 5 \quad [\times x] \\ \Rightarrow &x^4 - 36 = 5x^2 \\ \Rightarrow &x^4 - 5x^2 - 36 = 0 \\ \Rightarrow &(x^2 - 9)(x^2 + 4) = 0 \\ \Rightarrow &x^2 = 9 \text{ or } x^2 = -4\end{aligned}$$

But x is a real number, so $x^2 \neq -4$.

$$\text{Hence } x^2 = 9 \Rightarrow x = \pm 3$$

$$\text{Sub. } x = 3 \text{ in (2)} \Rightarrow y = \frac{-6}{3} = -2 \Rightarrow z = 3 - 2i,$$

$$\text{Sub. } x = -3 \text{ in (2)} \Rightarrow y = \frac{-6}{-3} = 2 \Rightarrow z = -3 + 2i$$

Hence $z = 3 - 2i$ or $z = -3 + 2i$.

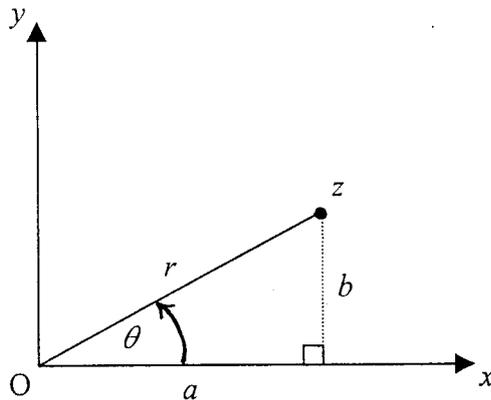
NOTES

1. You may verify that $(3 - 2i)^2 = 5 - 12i$ and $(-3 + 2i)^2 = 5 - 12i$.
2. $3 - 2i$ and $-3 + 2i$ are the square ^{roots} ~~roots~~ of the complex number $5 - 12i$.
3. There is another method for finding the square roots of a complex number (see later).

YOU MAY NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 1".

THE MODULUS AND ARGUMENT OF A COMPLEX NUMBER

In general, the complex number $z = a + bi$ is represented by the point with coordinates (a, b) on an **Argand diagram**.



The **modulus** of z is the distance Oz and is denoted by $|z|$ or r .

By Pythagoras: $r^2 = a^2 + b^2 \Rightarrow r = \sqrt{a^2 + b^2}$.

$$r = \sqrt{a^2 + b^2} \quad \text{or} \quad |z| = \sqrt{a^2 + b^2}$$

The **argument** of z is the angle between Oz and the **positive** direction of the x -axis.

The argument of z is denoted by $\arg(z)$ or θ and is such that $-180^\circ < \theta \leq 180^\circ$ (or $-\pi < \theta \leq \pi$ if θ is measured in radians).

It is not possible to give a general formula for calculating θ as the method of calculation differs according to the quadrant the complex number z lies in.

Note

$$\cos \theta = \frac{a}{r} \quad \Rightarrow \quad a = r \cos \theta$$

$$\sin \theta = \frac{b}{r} \quad \Rightarrow \quad b = r \sin \theta$$

$$\begin{aligned} \text{Now } z &= a + bi \\ &= r \cos \theta + (r \sin \theta)i \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

Expressing a complex number z in the form $r(\cos \theta + i \sin \theta)$ is known as expressing the complex number in **polar form**.

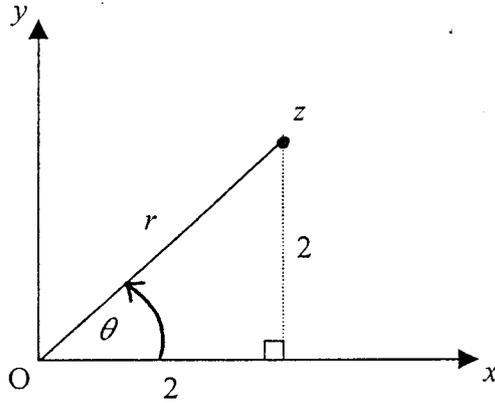
It is simple to multiply, divide and find powers of complex numbers when expressed in polar form (see later).

Worked Example 1 (First Quadrant)

Find the modulus and argument of the complex number $z = 2 + 2i$.
Hence express z in polar form.

Solution

$$z = 2 + 2i \quad \rightarrow \quad \text{plot } (2, 2)$$



$$r = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$\tan \theta = \frac{2}{2} = 1 \quad \Rightarrow \quad \theta = \tan^{-1} 1 = 45^\circ$$

$$\begin{aligned} \text{modulus} &= 2\sqrt{2} \\ \text{argument} &= 45^\circ \end{aligned}$$

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 2\sqrt{2}(\cos 45^\circ + i \sin 45^\circ) \end{aligned}$$

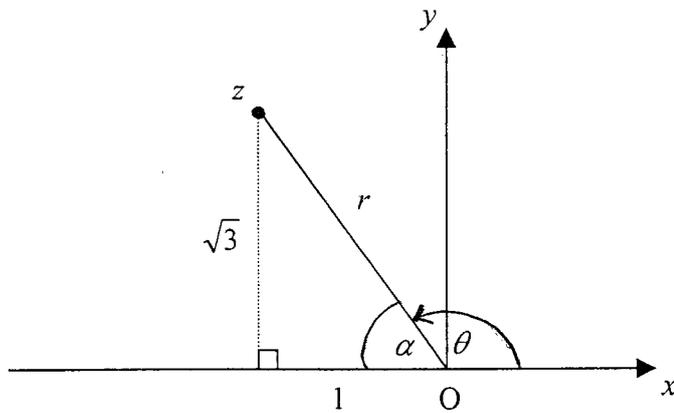
[Note that it is acceptable to give the argument in degrees unless the argument is specifically requested to be given in radians.]

Worked Example 2 (Second Quadrant)

Find the modulus and argument of the complex number $z = -1 + \sqrt{3}i$.
Hence express z in polar form.

Solution

$$z = -1 + \sqrt{3}i \rightarrow \text{plot } (-1, \sqrt{3})$$



$$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\begin{aligned} \tan \alpha &= \frac{\sqrt{3}}{1} = \sqrt{3} &\Rightarrow & \alpha = \tan^{-1} \sqrt{3} = 60^\circ \\ & &\Rightarrow & \theta = 180^\circ - 60^\circ = 120^\circ \end{aligned}$$

modulus = 2

argument = 120°

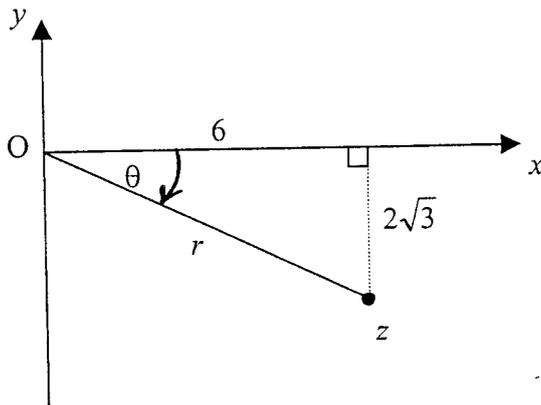
$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 2(\cos 120^\circ + i \sin 120^\circ) \end{aligned}$$

Worked Example 3 (Fourth Quadrant)

Find the modulus and argument of the complex number $z = 6 - 2\sqrt{3}i$.
Hence express z in polar form.

Solution

$$z = 6 - 2\sqrt{3}i \rightarrow \text{plot } (6, -2\sqrt{3})$$



$$r = \sqrt{6^2 + (2\sqrt{3})^2} = \sqrt{36 + 12} = \sqrt{48} = 4\sqrt{3}$$

Note that the argument θ must be such that $-180^\circ < \theta \leq 180^\circ$.

$$\begin{aligned} \tan \theta &= \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} &\Rightarrow & \theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30^\circ \\ & &\Rightarrow & \theta = -30^\circ \end{aligned}$$

[Strictly speaking, this is abuse of notation, however it ^{is} a convenient method of calculating the argument when the complex number lies in the fourth quadrant.]

$$\begin{aligned} \text{modulus} &= 4\sqrt{3} \\ \text{argument} &= -30^\circ \end{aligned}$$

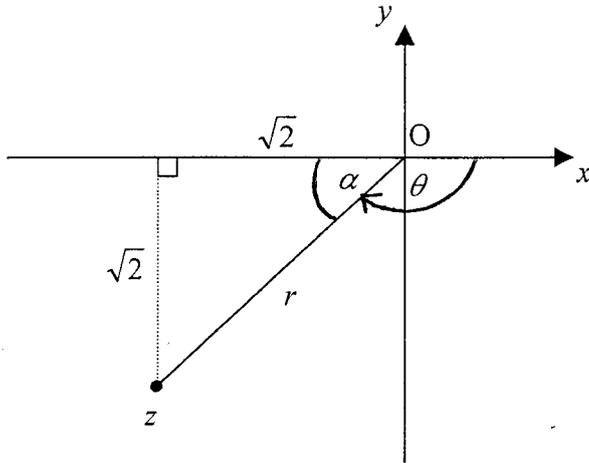
$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 4\sqrt{3}\{\cos(-30^\circ) + i \sin(-30^\circ)\} \end{aligned}$$

Worked Example 4 (Third Quadrant)

Find the modulus and argument of the complex number $z = -\sqrt{2}(1 + i)$.
Hence express z in polar form.

Solution

$$z = -\sqrt{2}(1 + i) = -\sqrt{2} - \sqrt{2}i \rightarrow \text{plot } (-\sqrt{2}, -\sqrt{2})$$



$$r = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2 + 2} = \sqrt{4} = 2$$

Note that the argument θ must be such that $-180^\circ < \theta \leq 180^\circ$.

$$\begin{aligned} \tan \alpha &= \frac{\sqrt{2}}{\sqrt{2}} = 1 & \Rightarrow & \alpha = \tan^{-1} 1 = 45^\circ \\ & & \Rightarrow & \theta = -180^\circ + 45^\circ = -135^\circ \end{aligned}$$

modulus = 2

argument = -135°

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 2\{\cos(-135^\circ) + i \sin(-135^\circ)\} \end{aligned}$$

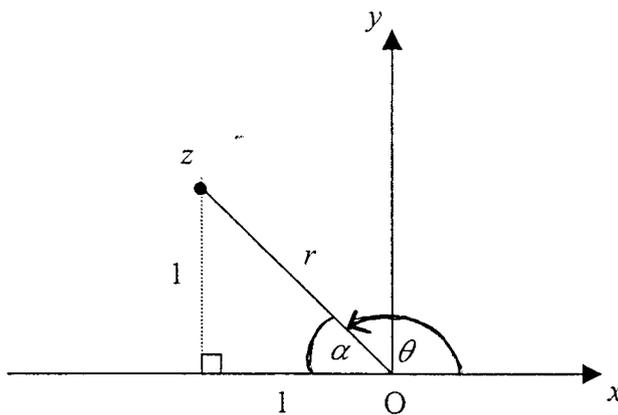
Worked Example 5

- (a) Express the complex number $\frac{1+3i}{1-2i}$ in the form $x+yi$, where x and y are real numbers.
(b) Hence find the modulus and argument of this complex number.

Solution

$$\begin{aligned} \text{(a)} \quad \frac{1+3i}{1-2i} &= \frac{(1+3i)(1+2i)}{(1-2i)(1+2i)} \\ &= \frac{1+5i+6i^2}{1-4i^2} \\ &= \frac{1+5i-6}{1+4} \\ &= \frac{-5+5i}{5} \\ &= -1+i \end{aligned}$$

$$\text{(b)} \quad \frac{1+3i}{1-2i} = -1+i \rightarrow \text{plot } (-1, 1)$$



$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan \alpha = \frac{1}{1} = 1 \Rightarrow \alpha = \tan^{-1} 1 = 45^\circ$$

$$\Rightarrow \theta = 180^\circ - 45^\circ = 135^\circ$$

$$\text{modulus} = \sqrt{2}$$

$$\text{argument} = 135^\circ$$

YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 2".

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS IN POLAR FORM

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be two general complex numbers expressed in polar form.

$$\begin{aligned}z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \\&= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\&= r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\&= r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 - \sin \theta_1 \sin \theta_2) \quad [\text{since } i^2 = -1] \\&= r_1 r_2 \{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\} \\&= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\}\end{aligned}$$

This means that to multiply two complex numbers in polar form:

- (1) multiply the moduli
- (2) add the arguments

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \\&= \frac{r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_2(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \quad *** \\&= \frac{r_1(\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 - i^2 \sin \theta_1 \sin \theta_2)}{r_2(\cos^2 \theta_2 - i^2 \sin^2 \theta_2)} \\&= \frac{r_1(\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + \sin \theta_1 \sin \theta_2)}{r_2(\cos^2 \theta_2 + \sin^2 \theta_2)} \quad [\text{since } i^2 = -1] \\&= \frac{r_1\{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)\}}{r_2(1)} \quad [\text{since } \cos^2 \theta_2 + \sin^2 \theta_2 = 1] \\&= \frac{r_1\{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\}}{r_2} \\&= \frac{r_1}{r_2} \{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\}\end{aligned}$$

*** multiplying both the numerator and denominator by the complex conjugate of $\cos \theta_2 + i \sin \theta_2$

This means that to divide two complex numbers in polar form:

- (1) divide the moduli
- (2) subtract the arguments

Worked Example

Let $z = 8(\cos 50^\circ + i \sin 50^\circ)$ and $w = 2(\cos 30^\circ + i \sin 30^\circ)$.

Express in the form $r(\cos \theta + i \sin \theta)$: (a) zw (b) $\frac{z}{w}$ (c) $\frac{w^3}{z^2}$

Solution

$$\begin{aligned} \text{(a)} \quad zw &= 8(\cos 50^\circ + i \sin 50^\circ) \cdot 2(\cos 30^\circ + i \sin 30^\circ) \\ &= 16(\cos 80^\circ + i \sin 80^\circ) \quad [\text{multiply the moduli and add the arguments}] \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{z}{w} &= \frac{8(\cos 50^\circ + i \sin 50^\circ)}{2(\cos 30^\circ + i \sin 30^\circ)} \\ &= 4(\cos 20^\circ + i \sin 20^\circ) \quad [\text{divide the moduli and subtract the arguments}] \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad w^3 &= www \\ &= 2(\cos 30^\circ + i \sin 30^\circ) \cdot 2(\cos 30^\circ + i \sin 30^\circ) \cdot 2(\cos 30^\circ + i \sin 30^\circ) \\ &= 8(\cos 90^\circ + i \sin 90^\circ) \quad [\text{multiply the moduli and add the arguments}] \end{aligned}$$

$$\begin{aligned} z^2 &= zz \\ &= 8(\cos 50^\circ + i \sin 50^\circ) \cdot 8(\cos 50^\circ + i \sin 50^\circ) \\ &= 64(\cos 100^\circ + i \sin 100^\circ) \quad [\text{multiply the moduli and add the arguments}] \end{aligned}$$

$$\begin{aligned} \frac{z^3}{w^2} &= \frac{8(\cos 90^\circ + i \sin 90^\circ)}{64(\cos 100^\circ + i \sin 100^\circ)} \\ &= \frac{1}{8} \{ \cos(-10^\circ) + i \sin(-10^\circ) \} \quad [\text{divide the moduli and subtract the arguments}] \end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 3".

DE MOIVRE'S THEOREM

Let $z = r(\cos \theta + i \sin \theta)$ be a general complex number expressed in polar form.

$$\begin{aligned} z^2 &= z z \\ &= r(\cos \theta + i \sin \theta) \cdot r(\cos \theta + i \sin \theta) \\ &= r^2(\cos 2\theta + i \sin 2\theta) \quad [\text{multiply the moduli and add the arguments}] \end{aligned}$$

$$\begin{aligned} z^3 &= z^2 z \\ &= r^2(\cos 2\theta + i \sin 2\theta) \cdot r(\cos \theta + i \sin \theta) \\ &= r^3(\cos 3\theta + i \sin 3\theta) \quad [\text{multiply the moduli and add the arguments}] \end{aligned}$$

$$\begin{aligned} z^4 &= z^3 z \\ &= r^3(\cos 3\theta + i \sin 3\theta) \cdot r(\cos \theta + i \sin \theta) \\ &= r^4(\cos 4\theta + i \sin 4\theta) \quad [\text{multiply the moduli and add the arguments}] \end{aligned}$$

In general:

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

This result for the power of a complex number in polar form is known as **de Moivre's Theorem**. It can be shown that de Moivre's theorem is also valid for negative and fractional powers of complex numbers in polar form.

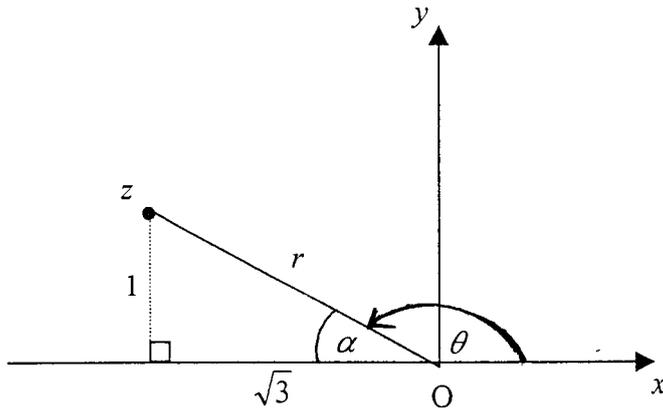
Worked Example 1

Write the complex number $z = -\sqrt{3} + i$ in polar form.

- Hence:
- (a) express z^4 in the form $x + yi$, where x and y are real numbers
 - (b) show that $z^6 + 64 = 0$.

Solution

$$z = -\sqrt{3} + i \rightarrow \text{plot } (-\sqrt{3}, 1)$$



$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\begin{aligned} \tan \alpha &= \frac{1}{\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ \\ &\Rightarrow \theta = 180^\circ - 30^\circ = 150^\circ \end{aligned}$$

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) \\ &= 2(\cos 150^\circ + i \sin 150^\circ) \end{aligned}$$

(a) By de Moivre's theorem:

$$\begin{aligned} z^4 &= 2^4 \{ \cos(4 \times 150^\circ) + i \sin(4 \times 150^\circ) \} \\ &= 16(\cos 600^\circ + i \sin 600^\circ) \\ &= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -8 - 8\sqrt{3}i \end{aligned}$$

(b) By de Moivre's theorem:

$$\begin{aligned} z^6 &= 2^6 \{ \cos(6 \times 150^\circ) + i \sin(6 \times 150^\circ) \} \\ &= 64(\cos 900^\circ + i \sin 900^\circ) \\ &= 64(-1 + 0i) \\ &= -64 \end{aligned}$$

$$\text{Hence } z^4 + 64 = -64 + 64 = 0.$$

[It is worth noting, in terms of exact values, that $\frac{\sqrt{3}}{2} = 0.866\dots$ and $\frac{1}{\sqrt{2}} = 0.707\dots$]

Worked Example 2

Let $z = 1 + \sqrt{3}i$ and $w = -2\sqrt{3} + 2i$.

- (a) Write z and w in polar form.
(b) Hence express in the form $x + yi$, where x and y are real numbers:

(i) $z^3 w^2$ (ii) $\frac{z^9}{w^3}$

Solution

(a) It can easily be shown that $z = 2(\cos 60^\circ + i \sin 60^\circ)$ and $w = 4(\cos 150^\circ + i \sin 150^\circ)$.

(b)(i) By de Moivre's theorem: $z^3 = 2^3 \{\cos(3 \times 60^\circ) + i \sin(3 \times 60^\circ)\}$
 $= 8(\cos 180^\circ + i \sin 180^\circ)$

$$w^2 = 4^2 \{\cos(2 \times 150^\circ) + i \sin(2 \times 150^\circ)\}$$
$$= 16(\cos 300^\circ + i \sin 300^\circ)$$

Hence $z^3 w^2 = 8(\cos 180^\circ + i \sin 180^\circ) \cdot 16(\cos 300^\circ + i \sin 300^\circ)$
 $= 128(\cos 480^\circ + i \sin 480^\circ)$ [multiply the moduli and add the arguments]
 $= 128 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)$
 $= -64 + 64\sqrt{3}i$

[Note that it is best to multiply z^3 and w^2 in polar form.]

(ii) By de Moivre's theorem: $z^9 = 2^9 \{\cos(9 \times 60^\circ) + i \sin(9 \times 60^\circ)\}$
 $= 512(\cos 540^\circ + i \sin 540^\circ)$

$$w^3 = 4^3 \{\cos(3 \times 150^\circ) + i \sin(3 \times 150^\circ)\}$$
$$= 64(\cos 450^\circ + i \sin 450^\circ)$$

Hence $\frac{z^9}{w^3} = \frac{512(\cos 540^\circ + i \sin 540^\circ)}{64(\cos 450^\circ + i \sin 450^\circ)}$
 $= 8(\cos 90^\circ + i \sin 90^\circ)$ [divide the moduli and subtract the arguments]
 $= 8(0 + 1i)$
 $= 8i$

[Note that it is best to divide z^9 and w^3 in polar form.]

YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 4".

ROOTS OF COMPLEX NUMBERS

Given any complex number w , it can be shown that there are **two square roots** of w . That is, there are two complex numbers z for which $z^2 = w$.

It can also be shown that there are **three cube roots** of w . That is, there are three complex numbers z for which $z^3 = w$.

In general, for any positive integer n , there are n complex numbers z for which $z^n = w$.

The following examples illustrate the method of finding the roots of a given complex number.

Worked Example 1

- (a) Write the complex number $2 + 2\sqrt{3}i$ in polar form.
(b) Find the two complex numbers z for which $z^2 = 2 + 2\sqrt{3}i$, expressing each root in the form $r(\cos \theta + i \sin \theta)$.
(c) Show the two roots on a single Argand diagram.

Solution

(a) It can easily be shown that $2 + 2\sqrt{3}i = 4(\cos 60^\circ + i \sin 60^\circ)$.

(b) $z^2 = 2 + 2\sqrt{3}i \quad \Rightarrow \quad z^2 = 4(\cos 60^\circ + i \sin 60^\circ) \quad \dots(*)$

First Root:
$$\begin{aligned} z &= \{4(\cos 60^\circ + i \sin 60^\circ)\}^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \left\{ \cos\left(\frac{1}{2} \times 60^\circ\right) + i \sin\left(\frac{1}{2} \times 60^\circ\right) \right\} \quad [\text{by de Moivre's theorem}] \\ &= 2(\cos 30^\circ + i \sin 30^\circ) \end{aligned}$$

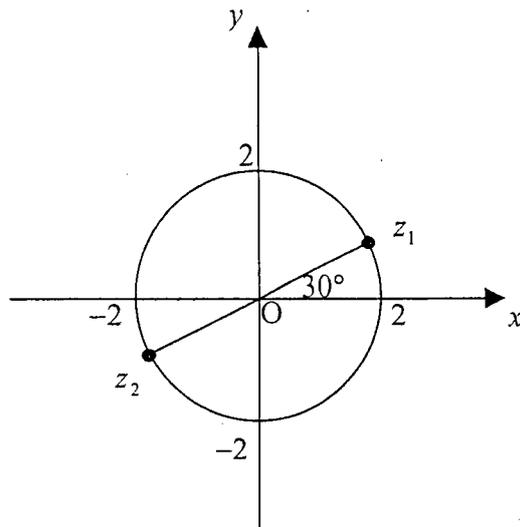
Second Roots : Equation (*) can be written as

$$\begin{aligned} z^2 &= 4\{\cos(60^\circ + 360^\circ) + i \sin(60^\circ + 360^\circ)\} \\ \Rightarrow z^2 &= 4(\cos 420^\circ + i \sin 420^\circ) \quad [\text{since } \cos 420^\circ = \cos 60^\circ \text{ and} \\ &\quad \sin 420^\circ = \sin 60^\circ] \end{aligned}$$

$$\begin{aligned} \Rightarrow z &= \{4(\cos 420^\circ + i \sin 420^\circ)\}^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \left\{ \cos\left(\frac{1}{2} \times 420^\circ\right) + i \sin\left(\frac{1}{2} \times 420^\circ\right) \right\} \\ &= 2(\cos 210^\circ + i \sin 210^\circ) \end{aligned}$$

Summary of Roots: $z_1 = 2(\cos 30^\circ + i \sin 30^\circ)$
 $z_2 = 2(\cos 210^\circ + i \sin 210^\circ)$

(c)



[Note that on an Argand diagram the roots lie diametrically opposite each other on the circle with centre O and radius 2. In general, the roots of a complex number always lie equally spaced on the circumference of a circle with centre O. If one root of a complex number can be found, this fact can therefore be used to find the other roots.]

Worked Example 2

- (a) Write the complex number $-8 + 8\sqrt{3}i$ in polar form.
(b) Find the two complex numbers z for which $z^2 = -8 + 8\sqrt{3}i$, expressing each root in the form $r(\cos \theta + i \sin \theta)$.
(c) Show the two roots on a single Argand diagram.

Solution

(a) It can easily be shown that $-8 + 8\sqrt{3}i = 16(\cos 120^\circ + i \sin 120^\circ)$.

(b) $z^2 = -8 + 8\sqrt{3}i \quad \Rightarrow \quad z^2 = 16(\cos 120^\circ + i \sin 120^\circ)$

First Root:

$$\begin{aligned} z &= \{16(\cos 120^\circ + i \sin 120^\circ)\}^{\frac{1}{2}} \\ &= 16^{\frac{1}{2}} \left\{ \cos\left(\frac{1}{2} \times 120^\circ\right) + i \sin\left(\frac{1}{2} \times 120^\circ\right) \right\} \text{ [by de Moivre's theorem]} \\ &= 4(\cos 60^\circ + i \sin 60^\circ) \end{aligned}$$

Second Root: The second root will lie diametrically opposite the first root on the circumference of the circle with centre O and radius 4.

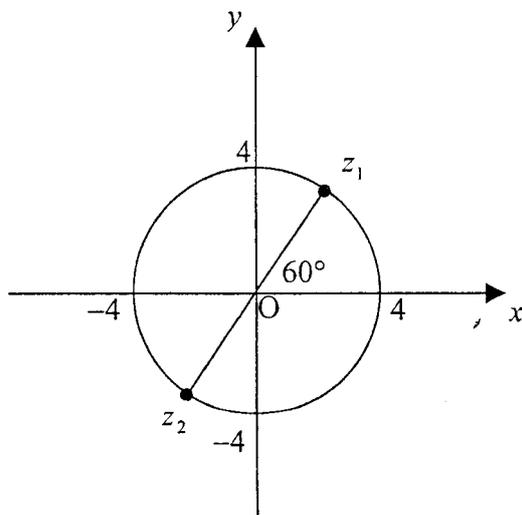
$$\frac{360^\circ}{2} = 180^\circ$$

$$\begin{aligned} z &= 4\{\cos(60^\circ + 180^\circ) + i \sin(60^\circ + 180^\circ)\} \\ &= 4(\cos 240^\circ + i \sin 240^\circ) \end{aligned}$$

Summary of Roots:

$$\begin{aligned} z_1 &= 4(\cos 60^\circ + i \sin 60^\circ) \\ z_2 &= 4(\cos 240^\circ + i \sin 240^\circ) \end{aligned}$$

(c)



Worked Example 3

Find the three cube roots of the complex number $8i$, expressing each root in the form $x + yi$, where x and y are real numbers. Show all three roots on a single Argand diagram.

Solution

The three cube roots of $8i$ are the three roots of the equation $z^3 = 8i$.

We must now write the complex number $8i$ in polar form.
It can easily be shown that $8i = 8(\cos 90^\circ + i \sin 90^\circ)$.

$$z^3 = 8i \quad \Rightarrow \quad z^3 = 8(\cos 90^\circ + i \sin 90^\circ)$$

$$\begin{aligned} \text{First Root:} \quad z &= \{8(\cos 90^\circ + i \sin 90^\circ)\}^{\frac{1}{3}} \\ &= 8^{\frac{1}{3}} \left\{ \cos\left(\frac{1}{3} \times 90^\circ\right) + i \sin\left(\frac{1}{3} \times 90^\circ\right) \right\} \quad [\text{by de Moivre's theorem}] \\ &= 2(\cos 30^\circ + i \sin 30^\circ) \end{aligned}$$

Second Root: The three roots will lie equally spaced on the circumference of the circle with centre O and radius 2.

$$\frac{360^\circ}{3} = 120^\circ$$

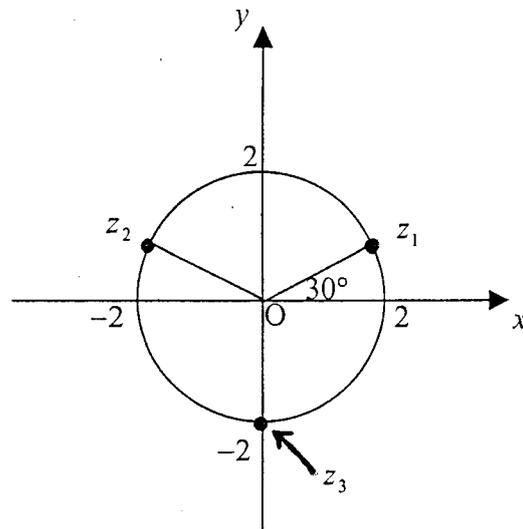
$$\begin{aligned} z &= 2\{\cos(30^\circ + 120^\circ) + i \sin(30^\circ + 120^\circ)\} \\ &= 2(\cos 150^\circ + i \sin 150^\circ) \end{aligned}$$

$$\begin{aligned} \text{Third Root:} \quad z &= 2\{\cos(150^\circ + 120^\circ) + i \sin(150^\circ + 120^\circ)\} \\ &= 2(\cos 270^\circ + i \sin 270^\circ) \end{aligned}$$

$$\text{Summary of Roots:} \quad z_1 = 2(\cos 30^\circ + i \sin 30^\circ) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$z_2 = 2(\cos 150^\circ + i \sin 150^\circ) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$z_3 = 2(\cos 270^\circ + i \sin 270^\circ) = 2(0 - 1i) = -2i$$



YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 5".

POLYNOMIAL EQUATIONS

Worked Example 1

Verify that $z = 2$ is a root of the equation $z^3 - 4z^2 + 9z - 10 = 0$.
Hence find all the roots of this equation.

Solution

Let $f(z) = z^3 - 4z^2 + 9z - 10$.

A real root can be tested using synthetic division (as in Higher).

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 9 & -10 \\ & & 2 & -4 & 10 \\ \hline & 1 & -2 & 5 & 0 \end{array}$$

Remainder = 0, hence $z = 2$ is a root of the equation.

The polynomial can now be factorised: $f(z) = (z - 2)(z^2 - 2z + 5)$

The remaining roots of the equation $f(z) = 0$ come from the quadratic equation $z^2 - 2z + 5 = 0$.

This equation must be solved using the quadratic formula with $a = 1$, $b = -2$ and $c = 5$:

$$\begin{aligned} z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \end{aligned}$$

Summary of Roots:

$$\begin{aligned} z &= 2 \\ z &= 1 + 2i \\ z &= 1 - 2i \end{aligned}$$

Let $f(z)$ be a polynomial in z with **real coefficients**.

It can be shown in general that the roots of the equation $f(z) = 0$ always occur in **conjugate pairs**.

That is, if $z = \alpha$ is a root of the equation $f(z) = 0$, then $z = \bar{\alpha}$ will also be a root.

Illustration

The four roots of the equation $z^4 - 2z^3 - z^2 + 2z + 10 = 0$ are $z = 2 + i$, $z = 2 - i$, $z = -1 + i$ and $z = -1 - i$. The four roots are in conjugate pairs.

NOTES

- (1) If one root of a polynomial equation can be found (where the polynomial has real coefficients), this fact can be used to find other roots of the equation.
- (2) The polynomial **must** have **real coefficients** for the roots to occur in conjugate pairs. For example, the roots of the equation $z^2 + (i - 2)z + 3 - i = 0$ are $z = 1 + i$ and $z = 1 - 2i$, which is not a conjugate pair.

Worked Example 2

- (a) Verify that $z = 1 + 2i$ is a root of the equation $z^4 - 6z^3 + 18z^2 - 30z + 25 = 0$.
(b) Write down another root of this equation.
(c) Find **all** the roots of the equation.

Solution

(a) $z = 1 + 2i$

$$\begin{array}{lll} z^2 = zz & z^3 = zz^2 & z^4 = zz^3 \\ = (1 + 2i)(1 + 2i) & = (1 + 2i)(-3 + 4i) & = (1 + 2i)(-11 - 2i) \\ = 1 + 4i + 4i^2 & = -3 - 2i + 8i^2 & = -11 - 24i - 4i^2 \\ = 1 + 4i - 4 & = -3 - 2i - 8 & = -11 - 24i + 4 \\ = -3 + 4i & = -11 - 2i & = -7 - 24i \end{array}$$

[Note that z^4 can also be found using $z^4 = z^2 z^2$.]

$$\begin{aligned} & z^4 - 6z^3 + 18z^2 - 30z + 25 \\ &= -7 - 24i - 6(-11 - 2i) + 18(-3 + 4i) - 30(1 + 2i) + 25 \\ &= -7 - 24i + 66 + 12i - 54 + 72i - 30 - 60i + 25 \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

Hence $z = 1 + 2i$ is a root of the equation $z^4 - 6z^3 + 18z^2 - 30z + 25 = 0$.

- (b) The conjugate $z = 1 - 2i$ is also a root of the equation, as the polynomial has real coefficients.
(c) Let $f(z) = z^4 - 6z^3 + 18z^2 - 30z + 25$.

Recall that if $z = \alpha$ is a root of the equation $f(z) = 0$, then $(z - \alpha)$ is a factor of $f(z)$. This fact can be used to find two factors of $f(z)$ from the two roots known.

<i>Root</i>		<i>Factor</i>
$z = 1 + 2i$	\rightarrow	$z - (1 + 2i)$
$z = 1 - 2i$	\rightarrow	$z - (1 - 2i)$

These factors can now be multiplied together to form a quadratic factor of $f(z)$:

$$\begin{aligned} \{z - (1 + 2i)\}\{z - (1 - 2i)\} &= \{(z - 1) - 2i\}\{(z - 1) + 2i\} \\ &= (z - 1)^2 - 4i^2 \\ &= z^2 - 2z + 1 + 4 \\ &= z^2 - 2z + 5 \end{aligned}$$

$f(z)$ can now be factorised using algebraic long division.

$$\begin{array}{r}
 z^2 - 2z + 5 \overline{) \begin{array}{r} z^4 - 6z^3 + 18z^2 - 30z + 25 \\ z^4 - 2z^3 + 5z^2 \\ \hline -4z^3 + 13z^2 - 30z + 25 \\ -4z^3 + 8z^2 - 20z \\ \hline 5z^2 - 10z + 25 \\ 5z^2 - 10z + 25 \\ \hline 0 \end{array}}
 \end{array}$$

[Note that the remainder should be zero at this stage, since we know that $z^2 - 2z + 5$ is a factor of $f(z)$.]

$f(z)$ can now be factorised: $f(z) = (z^2 - 2z + 5)(z^2 - 4z + 5)$

The remaining roots of the equation $f(z) = 0$ come from the quadratic equation $z^2 - 4z + 5 = 0$.

This equation must be solved using the quadratic formula with $a = 1$, $b = -4$ and $c = 5$:

$$\begin{aligned}
 z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2 \times 1} \\
 &= \frac{4 \pm \sqrt{-4}}{2} \\
 &= \frac{4 \pm 2i}{2} \\
 &= 2 \pm i
 \end{aligned}$$

Summary of Roots:

$$\begin{aligned}
 z &= 1 + 2i \\
 z &= 1 - 2i \\
 z &= 2 + i \\
 z &= 2 - i
 \end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 6".

TRIGONOMETRIC IDENTITIES

A complex number method can be used to find identities for $\cos n\theta$ and $\sin n\theta$ in terms of $\cos \theta$ and $\sin \theta$.

First note the pattern of powers of i :

$$i^0 = 1$$

$$i^2 = -1$$

$$i^3 = i^2 i = (-1)i = -i$$

$$i^4 = i^3 i = (-i)i = -i^2 = 1$$

$$i^5 = i^4 i = 1i = i$$

$$i^6 = i^5 i = ii = i^2 = -1$$

and so on

The cyclic pattern $1, i, -1, -i$ repeats indefinitely for powers of i .

If $z = r(\cos \theta + i \sin \theta)$, recall that de Moivre's theorem states that $z^n = r^n (\cos n\theta + i \sin n\theta)$ for any integer n . In particular, when $r = 1$, this gives:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Worked Example 1

Starting with

$$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5,$$

find identities for $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos \theta$ and $\sin \theta$
Hence find an identity for $\tan 5\theta$ entirely in terms of $\tan \theta$.

Solution

$$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5$$

Now expand $(\cos \theta + i \sin \theta)^5$ using the binomial theorem:

$$\begin{aligned} \cos 5\theta + i \sin 5\theta &= (\cos \theta)^5 + 5 \cdot (\cos \theta)^4 \cdot i \sin \theta + 10 \cdot (\cos \theta)^3 \cdot (i \sin \theta)^2 + 10 \cdot (\cos \theta)^2 \cdot (i \sin \theta)^3 \\ &\quad + 5 \cdot \cos \theta \cdot (i \sin \theta)^4 + (i \sin \theta)^5 \\ &= \cos^5 \theta + 5 \cos^4 \theta \cdot i \sin \theta + 10 \cos^3 \theta \cdot i^2 \sin^2 \theta + 10 \cos^2 \theta \cdot i^3 \sin^3 \theta + 5 \cos \theta \cdot i^4 \sin^4 \theta \\ &\quad + i^5 \sin^5 \theta \end{aligned}$$

Now replace the powers of i :

$$\cos 5\theta + i \sin 5\theta = \cos^5 \theta + 5 \cos^4 \theta \cdot i \sin \theta + 10 \cos^3 \theta \cdot (-1) \sin^2 \theta + 10 \cos^2 \theta \cdot (-i) \sin^3 \theta + 5 \cos \theta \cdot 1 \sin^4 \theta + i \sin^5 \theta$$

$$\text{Equating real parts} \quad \Rightarrow \quad \cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$

$$\text{Equating imaginary parts} \quad \Rightarrow \quad \sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$= \frac{5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta}{\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta} \quad [\text{now divide all terms by } \cos^5 \theta]$$

$$= \frac{\frac{5 \cos^4 \theta \sin \theta}{\cos^5 \theta} - \frac{10 \cos^2 \theta \sin^3 \theta}{\cos^5 \theta} + \frac{\sin^5 \theta}{\cos^5 \theta}}{\frac{\cos^5 \theta}{\cos^5 \theta} - \frac{10 \cos^3 \theta \sin^2 \theta}{\cos^5 \theta} + \frac{5 \cos \theta \sin^4 \theta}{\cos^5 \theta}}$$

$$= \frac{\frac{5 \sin \theta}{\cos \theta} - \frac{10 \sin^3 \theta}{\cos^3 \theta} + \frac{\sin^5 \theta}{\cos^5 \theta}}{1 - \frac{10 \sin^2 \theta}{\cos^2 \theta} + \frac{5 \sin^4 \theta}{\cos^4 \theta}}$$

$$= \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 7".

A complex number method can also be used to find identities for $\cos^n \theta$ and $\sin^n \theta$.

Let $z = \cos \theta + i \sin \theta$.

Then de Moivre's theorem states that $z^n = \cos n\theta + i \sin n\theta$.

$$\begin{aligned} \frac{1}{z^n} &= \frac{1}{\cos n\theta + i \sin n\theta} \\ &= \frac{1(\cos n\theta - i \sin n\theta)}{(\cos n\theta + i \sin n\theta)(\cos n\theta - i \sin n\theta)} \quad *** \\ &= \frac{\cos n\theta - i \sin n\theta}{\cos^2 n\theta - i^2 \sin^2 n\theta} \\ &= \frac{\cos n\theta - i \sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} \quad [\text{since } i^2 = -1] \\ &= \frac{\cos n\theta - i \sin n\theta}{1} \quad [\text{since } \cos^2 n\theta + \sin^2 n\theta = 1] \\ &= \cos n\theta - i \sin n\theta \end{aligned}$$

*** multiplying both the numerator and denominator by the complex conjugate of $\cos n\theta + i \sin n\theta$

[Alternatively: $\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i \sin(-n\theta)$ by de Moivre's theorem
 $= \cos n\theta - i \sin n\theta$ since $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$]

Hence $z^n + \frac{1}{z^n} = (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta)$
 $= 2 \cos n\theta$

In particular, when $n = 1$: $z + \frac{1}{z} = 2 \cos \theta$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$
$$z + \frac{1}{z} = 2 \cos \theta$$

Worked Example 2

Let $z = \cos \theta + i \sin \theta$.

Starting with $(2 \cos \theta)^4 = \left(z + \frac{1}{z}\right)^4$, show that $\cos^4 \theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}$.

Hence find $\int \cos^4 \theta d\theta$.

Solution

$$(2 \cos \theta)^4 = \left(z + \frac{1}{z}\right)^4$$

Now $(2 \cos \theta)^4 = 16 \cos^4 \theta$ and expand $\left(z + \frac{1}{z}\right)^4$ using the binomial theorem:

$$\begin{aligned} 16 \cos^4 \theta &= z^4 + 4 \cdot z^3 \cdot \frac{1}{z} + 6 \cdot z^2 \cdot \left(\frac{1}{z}\right)^2 + 4 \cdot z \cdot \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4 \\ &= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} \\ &= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 \\ &= 2 \cos 4\theta + 4(2 \cos 2\theta) + 6 \quad \left[\text{using the result } z^n + \frac{1}{z^n} = 2 \cos n\theta\right] \\ &= 2 \cos 4\theta + 8 \cos 2\theta + 6 \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos^4 \theta &= \frac{1}{16} (2 \cos 4\theta + 8 \cos 2\theta + 6) \\ &= \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8} \end{aligned}$$

$$\begin{aligned} \int \cos^4 \theta d\theta &= \int \left(\frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}\right) d\theta \\ &= \frac{1}{8} \cdot \frac{1}{4} \sin 4\theta + \frac{1}{2} \cdot \frac{1}{2} \sin 2\theta + \frac{3}{8} \theta + C \\ &= \frac{1}{32} \sin 4\theta + \frac{1}{4} \sin 2\theta + \frac{3}{8} \theta + C \end{aligned}$$

Recall that if $z = \cos \theta + i \sin \theta$, then $z^n = \cos n\theta + i \sin n\theta$ and $\frac{1}{z^n} = \cos n\theta - i \sin n\theta$.

$$\begin{aligned} z^n - \frac{1}{z^n} &= (\cos n\theta + i \sin n\theta) - (\cos n\theta - i \sin n\theta) \\ &= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta \\ &= 2i \sin n\theta \end{aligned}$$

In particular, when $n=1$: $z - \frac{1}{z} = 2i \sin \theta$

$$\begin{aligned} z^n - \frac{1}{z^n} &= 2i \sin n\theta \\ z - \frac{1}{z} &= 2i \sin \theta \end{aligned}$$

Worked Example 3

Let $z = \cos \theta + i \sin \theta$.

Starting with $(2i \sin \theta)^3 = \left(z - \frac{1}{z}\right)^3$, show that $\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$.

Hence find $\int \sin^3 \theta d\theta$.

Solution

$$(2i \sin \theta)^3 = \left(z - \frac{1}{z}\right)^3$$

Now $(2i \sin \theta)^3 = 8i^3 \sin^3 \theta = 8(-i) \sin^3 \theta = -8i \sin^3 \theta$ and expand $\left(z - \frac{1}{z}\right)^3$ using the binomial theorem:

$$\begin{aligned} -8i \sin^3 \theta &= z^3 + 3 \cdot z^2 \cdot \left(-\frac{1}{z}\right) + 3 \cdot z \cdot \left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3 \\ &= z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \\ &= \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right) \\ &= 2i \sin 3\theta - 3(2i \sin \theta) \quad \left[\text{using the result } z^n - \frac{1}{z^n} = 2i \sin n\theta\right] \\ &= 2i \sin 3\theta - 6i \sin \theta \end{aligned}$$

$$\Rightarrow -8 \sin^3 \theta = 2 \sin 3\theta - 6 \sin \theta \quad [\text{dividing through by } i]$$

$$\Rightarrow \sin^3 \theta = -\frac{1}{8}(2 \sin 3\theta - 6 \sin \theta)$$

$$= -\frac{1}{4} \sin 3\theta + \frac{3}{4} \sin \theta$$

$$= \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$$

$$\begin{aligned} \int \sin^3 \theta d\theta &= \int \left(\frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta \right) d\theta \\ &= \frac{3}{4}(-\cos \theta) - \frac{1}{4} \left(-\frac{1}{3} \cos 3\theta \right) + C \\ &= -\frac{3}{4} \cos \theta + \frac{1}{12} \cos 3\theta + C \end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "COMPLEX NUMBERS 8".

LOCUS IN THE COMPLEX PLANE

Suppose a complex number z moves in the complex plane subject to some constraint (for example, that $|z| = 3$ or $\arg(z) = \frac{\pi}{3}$).

The path of the complex number z is known as the **locus** of z . The equation of the locus of z can be found.

Recall that the modulus of the complex number $a + bi$ is given by the formula $\sqrt{a^2 + b^2}$. This can be written as:

$$|a + bi| = \sqrt{a^2 + b^2}$$

This formula for the modulus of a complex number can be used to find the equation of the locus of z .

Recall from Higher that the equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

Worked Example 1

The complex number z moves in the complex plane subject to the condition $|z| = 3$. Find the equation of the locus of z and interpret the locus geometrically.

Solution

Let $z = x + yi$. Then $|z| = \sqrt{x^2 + y^2}$.

$$\begin{aligned} |z| = 3 &\Rightarrow \sqrt{x^2 + y^2} = 3 \\ &\Rightarrow x^2 + y^2 = 9 \end{aligned}$$

The locus of z is the set of points on the circumference of the circle with centre O and radius 3.

Worked Example 2

The complex number z moves in the complex plane subject to the condition $|z + 1 - 2i| = 4$. Find the equation of the locus of z and interpret the locus geometrically.

Solution

Let $z = x + yi$.

$$\begin{aligned}z + 1 - 2i &= x + yi + 1 - 2i \\ &= (x + 1) + (y - 2)i\end{aligned}$$

$$|z + 1 - 2i| = \sqrt{(x + 1)^2 + (y - 2)^2}$$

$$\begin{aligned}|z + 1 - 2i| = 4 &\Rightarrow \sqrt{(x + 1)^2 + (y - 2)^2} = 4 \\ &\Rightarrow (x + 1)^2 + (y - 2)^2 = 16\end{aligned}$$

The locus of z is the set of points on the circumference of the circle with centre $(-1, 2)$ and radius 4.

Worked Example 3

The complex number z moves in the complex plane subject to the condition $|z + i| < 2$. Find the equation of the locus of z and interpret the locus geometrically.

Solution

Let $z = x + yi$.

$$\begin{aligned}z + i &= x + yi + i \\ &= x + (y + 1)i\end{aligned}$$

$$|z + i| = \sqrt{x^2 + (y + 1)^2}$$

$$\begin{aligned}|z + i| = 2 &\Rightarrow \sqrt{x^2 + (y + 1)^2} = 2 \\ &\Rightarrow x^2 + (y + 1)^2 = 4\end{aligned}$$

The locus of z is the set of points which lie inside the circle with centre $(0, -1)$ and radius 2.

[Note that if the condition was $|z + i| \leq 2$, the locus of z would be the set of points which lie on or inside the circle with centre O and radius 2.]

Worked Example 4

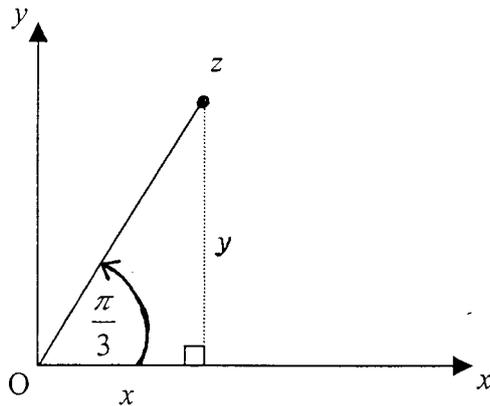
The complex number z moves in the complex plane subject to the condition $\arg(z) = \frac{\pi}{3}$.

Find the equation of the locus of z and interpret the locus geometrically.

Solution

Let $z = x + yi$.

The locus of z is the straight line below.



$$\begin{aligned}\tan \frac{\pi}{3} = \frac{y}{x} &\Rightarrow \frac{y}{x} = \sqrt{3} \\ &\Rightarrow y = \sqrt{3}x\end{aligned}$$

The locus of z is the part of the straight line with equation $y = \sqrt{3}x$ with $x > 0$.

[Note that a complex number which lies on the part of the line $y = \sqrt{3}x$ with $x < 0$ does not have an argument of $\frac{\pi}{3}$.]

Worked Example 5

The complex number z moves in the complex plane such that $|z + 2| = |z - i|$.

Show that the locus of z is a straight line and find the equation of the locus of z .

Solution

Let $z = x + yi$.

$$z + 2 = x + yi + 2$$

$$= (x + 2) + yi \Rightarrow |z + 2| = \sqrt{(x + 2)^2 + y^2}$$

$$z - i = x + yi - i$$

$$= x + (y - 1)i \Rightarrow |z - i| = \sqrt{x^2 + (y - 1)^2}$$

$$|z + 2| = |z - i| \Rightarrow \sqrt{(x + 2)^2 + y^2} = \sqrt{x^2 + (y - 1)^2}$$

$$\Rightarrow (x + 2)^2 + y^2 = x^2 + (y - 1)^2$$

$$\Rightarrow x^2 + 4x + 4 + y^2 = x^2 + y^2 - 2y + 1$$

$$\Rightarrow 4x + 4 = -2y + 1$$

$$\Rightarrow 4x + 2y + 3 = 0$$

This equation is of the form $Ax + By + C = 0$, hence the locus of z is a straight line.

The equation of the locus of z is $4x + 2y + 3 = 0$.

YOU CAN NOW ATTEMPT THE WORKSHEETS "COMPLEX NUMBERS 9" AND "COMPLEX NUMBERS 10".